Poynting's Theorem :→

By the Maxwell's Equations.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t} \quad \text{——— (a)}$$

$$\vec{\nabla} \times \vec{H} = J + \frac{\partial \vec{D}}{\partial t} \quad \text{——— (b)}$$

$$\vec{H} \cdot (\vec{\nabla} \times \vec{E}) = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \quad \text{——— (c)}$$

$$\vec{E} \cdot (\vec{\nabla} \times \vec{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \quad \text{——— (d)}$$

Substracting equation (d) from equation (c)

$$\vec{H} \cdot \vec{\nabla} \times \vec{E} - \vec{E} \cdot (\vec{\nabla} \times \vec{H}) = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \vec{J} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = -\vec{E} \cdot \vec{J} - \left(H \cdot \frac{\partial}{\partial t}(\mu \vec{H}) + \vec{E} \cdot \frac{\partial}{\partial t}(\epsilon \vec{E}) \right)$$

$$\left\{ \begin{array}{c} \because \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot (\vec{E} \times \vec{H}) \\ \vec{B} = \mu_0 \vec{H} \\ \vec{D} = \epsilon \vec{E} \end{array} \right\}$$

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = -\vec{E} \cdot \vec{J} - \left(\mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} + \epsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \right)$$

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = -\vec{E} \cdot \vec{J} - \left(\frac{1}{2} \mu \frac{\partial}{\partial t} H^2 + \frac{1}{2} \epsilon \frac{\partial E^2}{\partial t} \right)$$

$$\left\{ \frac{\partial}{\partial t} H^2 = 2 \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} \right\}$$

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = -\vec{E} \cdot \vec{J} - \frac{\partial}{\partial t} \left(\frac{1}{2} \mu H^2 + \frac{1}{2} \epsilon E^2 \right)$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right) = -\vec{\nabla}\cdot(\vec{E}\times\vec{H}) - \vec{E}\cdot\vec{J}$$

$U_E = \frac{1}{2}\epsilon E^2$ energy density in electric field.

$U_H = \frac{1}{2}\mu H^2$ energy density in magnetic field.

Taking volume integral on both side.

$$\int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right) dV = -\int \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dV - \int \vec{E}\cdot\vec{J}\,dV$$

using Gauss' Divergence Theorem

$$\int \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right) dV = -\int (\vec{E}\times\vec{H})\cdot d\vec{A} - \int \vec{E}\cdot\vec{J}\,dV$$

$$\boxed{\int \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right) dV = -\int \vec{S}\cdot d\vec{A} - \int \vec{E}\cdot\vec{J}\,dV}$$

Poynting Theorem

where $\boxed{\vec{S} = \vec{E}\times\vec{H}}$

$\vec{S}$ is known as poynting vector

(Energy Density.)

$U_E = \frac{1}{2}\epsilon E^2$

$U_H = \frac{1}{2}\epsilon H^2$

Since the time average of the square of a sine wave.

$\vec{E} = \vec{E_0} \sin(\omega t + \theta)$, $\vec{H} = \vec{H_0} \sin(\omega t + \theta)$

$\langle \sin^2 x \rangle = \frac{1}{2}$

Now the electromagnetic energy density

$\langle U_E \rangle = \frac{1}{2} \times \frac{1}{2} \epsilon E_0^2$

$\langle U_E \rangle = \frac{1}{4} \epsilon E_0^2$

Similarly $\langle U_H \rangle = \frac{1}{4} \mu H_0^2$

$\langle U \rangle = \langle U_E \rangle + \langle U_H \rangle$

$\langle U \rangle = \left(\frac{1}{4} \epsilon E_0^2 + \frac{1}{4} \mu H_0^2\right)$

Now the ratio of the time average energy densities in the electric and magnetic field in the electromagnetic wave is

$$\frac{\langle U_E \rangle}{\langle U_H \rangle} = \frac{\cancel{\frac{1}{4}} \epsilon E_0^2}{\cancel{\frac{1}{4}} \mu H_0^2}$$

$\vec{B} = \mu \vec{H}$

$$\frac{\langle U_E \rangle}{\langle U_H \rangle} = \frac{\epsilon E_0^2}{\cancel{\mu} \frac{B_0^2}{\mu^2}}$$

$$\frac{\langle U_E \rangle}{\langle U_H \rangle} = \mu \epsilon \frac{E_0^2}{B_0^2}$$

$$\frac{\langle U_E \rangle}{\langle U_H \rangle} = \mu \epsilon v^2$$

but $v = \frac{1}{\sqrt{\mu \epsilon}}$

$$\boxed{\frac{\langle U_E \rangle}{\langle U_H \rangle} = 1.}$$

Thus, the energy in an electromagnetic wave is shared equally between the electric and magnetic fields.